The Modified Weak Gap Condition of Real $G$-Modules

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$G$: a finite group. $V$: a real $G$-module

**Definition**

Let $H \in S(G)$ and $H < K \in S(G)$. $V$ satisfies the **gap condition** for $(H, K) \overset{\text{def}}{=} (GC)$

$$d_V(H, K) = \dim V^H - 2 \dim V^K > 0.$$  

**Definition**

Let $H \in S(G)$ and $H < K \in S(G)$. $V$ satisfies the **weak gap condition** for $(H, K) \overset{\text{def}}{=} (WC)$

$$d_V(H, K) = \dim V^H - 2 \dim V^K \geq 0.$$  

Let $\mathcal{F}$ be a set of subgroups of $G$ such that all minimal elements of $\mathcal{F}$ are normal in $G$.

Let

$$V^{\mathcal{F}} = \sum_{L \in \mathcal{F}} V^L = \sum_{L \in \text{min}({\mathcal{F}})} V^L,$$

$$V = V^{\mathcal{F}} \oplus V_{\mathcal{F}}$$

as real $G$-modules,

$$\mathcal{L}(G) = \{ L \in S(G) \mid L \supset G^\{r\} \}$$

for some $r \in \mathcal{P}$ (prime),

$$\mathcal{M}(G) = S(G) \setminus \mathcal{L}(G).$$
Theorem (L–M [1])

Let $V = \mathbb{R}[G]_{\mathcal{L}(G)}$. Then the following holds.

1. $V^H \neq 0 \iff H \in \mathcal{M}(G)$.

2. $V$ satisfies the weak gap condition for $(H, K)$ whenever $H < K \in \mathcal{S}(G)$.

3. Let $H \in \mathcal{M}(G)$ and $H < K \in \mathcal{S}(G)$. Then
   \[ \dim V^H = 2 \dim V^K \text{ holds } \iff \]
   \[ \begin{cases} |K : H| = 2, |KG^{(2)} : HG^{(2)}| = 2, \text{ and} \\ HG^r = G \text{ for all } r \in \mathcal{P}_{\text{odd}} \quad (\text{odd primes}). \end{cases} \]
Corollary

Let $G$ be an Oliver group. If $\exists$ at least two odd primes $r$ such that

$$G \neq G^\{r\},$$

then $\mathbb{R}[G]_{\mathcal{L}(G)}$ satisfies the gap condition for all $(H, K)$ such that $H \in \mathcal{M}(G)$ and $H < K \in S(G)$. 
Definition

Let $V$ be a real $G$-module. We say that $V$ satisfies the modified weak gap condition for $\mathcal{P}(G)$ if the following properties all are satisfied.

(M1) $V$ satisfies the weak gap condition for all $(P, H)$ such that $P \in \mathcal{P}(G)$ and $P < H \in S(G)$.

(M2) Let $P \in \mathcal{P}(G)$ and $P < H \in S(G)$. If $\dim V^P = 2 \dim V^H$ then $|H : P| = 2$.

(M3) Let $P \in \mathcal{P}(G)$, $P < H_1 \in S(G)$ and $P < H_2 \in S(G)$. If $\dim V^P = 2 \dim V^{H_1}$ and $\dim V^P = 2 \dim V^{H_2}$ then $\langle H_1, H_2 \rangle \in \mathcal{M}(G)$. 

Let $G$ be an Oliver group and $V$ a real $G$-module satisfying the modified weak gap condition for $\mathcal{P}(G)$. If $V$ satisfies the additional conditions below then $\exists$ smooth one-fixed-point $G$-action on the sphere $S^n$, where $n = \dim V$, such that $T_{x_0}(S^n) \cong V$ as real $G$-modules, where $x_0$ is the $G$-fixed point in $S^n$.

1. $\mathcal{G}^1(G) \subseteq \text{Iso}(G, V)$.
2. $\dim V^P \geq 5$ for all $P \in \mathcal{P}(G)$.
3. $\dim V^H \geq 3$ for all $H \in \mathcal{G}^1(G)$ (mod-\mathcal{P} cyclic subgroups).
4. If $P \in \mathcal{P}(G)$ and $\dim V^P = 2 \dim V^H$ then $\dim V^H - \dim V^{>H} > 1$.
5. $V$ is $\mathcal{L}(G)$-free, i.e. $V^H = 0$ for all $H \in \mathcal{L}(G)$. 
Reference


We have often used the next theorem.

**Theorem 1.2 ($\mathcal{L}(G)$-Regular Representation)**

*Let $G$ be an Oliver group. Then $V = \mathbb{R}[G]_{\mathcal{L}(G)}$ satisfies the modified weak gap condition for $\mathcal{P}(G)$.*

We would like to recall the proof of this theorem.
Proof. Let $P \in \mathcal{P}(G)$ and $P < H_1$, $P < H_2 \in S(G)$ such that
\[
dim V^P = 2 \dim V^{H_1} \text{ and } \dim V^P = 2 \dim V^{H_2}.
\]
We will show a contradiction under the assumption that
\[
\langle H_1, H_2 \rangle \in \mathcal{L}(G).
\]
L–M’s Theorem (of $\mathbb{R}[G]_{\mathcal{L}(G)}$) implies the following properties.

- $P \in \mathcal{P}(G)$,
- $|H_1 : P| = 2$ and $|H_1 G^{\{2\}} : PG^{\{2\}}| = 2$,
- $H_1 G^{\{q\}} = G$ for all $q \in \mathcal{P}_{\text{odd}}$,
- $|H_2 : P| = 2$ and $|H_2 G^{\{2\}} : PG^{\{2\}}| = 2$,
- $H_2 G^{\{q\}} = G$ for all $q \in \mathcal{P}_{\text{odd}}$. 
Set $\tilde{D} = \langle H_1, H_2 \rangle$.

Remark that $\tilde{D}/P$ is a dihedral group of order $\geq 4$.

**Case $\tilde{D} \supset G\{q\}$.** Suppose $\tilde{D} \supset G\{q\}$ for some $q \in P_{\text{odd}}$.

Since $H_1 G\{q\} = G$, we have

$$G = H_1 \tilde{D} = \tilde{D} \triangleright P.$$  

This contradicts that $G$ is an Oliver group.
Case $\tilde{D} \supset G^{\{2\}}$. Suppose $\tilde{D} \supset G^{\{2\}}$.

We have the sequence

$$P \triangleleft \tilde{C} \triangleleft \tilde{D} < G$$

such that $\tilde{C}/P$ is a cyclic group of order $\geq 2$ and $|\tilde{D}:\tilde{C}| = 2$.

Note that $N = \bigcap_{g \in G} g\tilde{C}g^{-1}$ is a normal subgroup of $G$, hence of $\tilde{D}$.

Since $|\tilde{D}:\tilde{C}| = 2$, we get $|\tilde{D}:N| = 2^a$ for some $a \in \mathbb{N}$.

Since $|G:\tilde{D}| = 2^b$ and $|\tilde{D}:N| = 2^a$ ($b \in \mathbb{N} \cup \{0\}$), we get

$$G^{\{2\}} \subset N \subset \tilde{C}.$$
Observe the homomorphism $G^{\{2\}} \rightarrow \tilde{C}/P$.

The kernel is $P \cap G^{\{2\}}$.

The sequence

$$(P \cap G^{\{2\}}) \triangleleft G^{\{2\}} \triangleleft G$$

contradicts that $G$ is an Oliver group.

Putting all this together, we conclude

$$\tilde{D} = \langle H_1, H_2 \rangle \in \mathcal{M}(G).$$
Lemma

Let $G$ be a finite group and $H$ and $K$ are normal subgroups of $G$ with 2-power indeces. Then $H \cap K$ is a normal subgroup of $G$ with 2-power index.

Proof. Observe the homomorphism $\varphi : H \rightarrow G/K$.

The kernel of $\varphi$ is $H \cap K$.

Since $\overline{\varphi} : H/(H \cap K) \rightarrow G/K$ is injective, the order of $H/(H \cap K)$ is a power of 2.

As $G/H$ is a 2-group, $G/(H \cap K)$ is a 2-group.
Theorem 3.1

Let $G$ be an Oliver group and let $Y$ be a disk with $G$-action such that $Y^G = \{a, b\}$ ($a \neq b$). Suppose the following (1)–(3).

1. $Y$ satisfy the weak gap condition for $\mathcal{P}(G)$, i.e.
   \[ \dim Y^P - 2 \dim Y^H \geq 0 \text{ for all } P \in \mathcal{P}(G) \text{ and } P < H \in S(G). \]
2. $Y^L \cap \partial Y = \emptyset$ for all $L \in \mathcal{L}(G)$.
3. For each $P \in \mathcal{P}(G)$, $Y^P$ is simply connected.

Then $T_a(Y) \oplus U$ and $T_b(Y) \oplus U$ are Smith equivalent for some $\mathcal{L}(G)$-free real $G$-module $U$. 
Theorem 3.2

Let $G$ be an Oliver group and let $Y$ be a disk with $G$-action such that $Y^G = \{a, b\}$ ($a \neq b$). Suppose the following (1)–(3).

1. $Y$ satisfy the gap condition for $\mathcal{P}(G)$, i.e.
   \[ \dim Y^P - 2 \dim Y^H > 0 \text{ for all } P \in \mathcal{P}(G) \text{ and } P < H \in S(G). \]

2. $Y^L \cap \partial Y = \emptyset$ for all $L \in \mathcal{L}(G)$.

3. For each $p \in \mathcal{P}$ and $P \in \mathcal{P}_p(G)$, $|\pi_1(Y^P)| < \infty$ and $\left( |\pi_1(Y^P)|, p \right) = 1$. (See Theorem 3.2 of [3] (2006).)

Then $T_a(Y) \oplus U$ and $T_b(Y) \oplus U$ are Smith equivalent for some $\mathcal{L}(G)$-free real $G$-module $U$. 

Theorem 3.2 of [3]. Let $\hat{G} = G \ltimes \pi$ be a finite group, $\zeta : \hat{G} \to G$ the canonical projection, $\hat{w} : \hat{G} \to \{1, -1\}$ and $w : G \to \{1, -1\}$ orientation homomorphisms with $\hat{w} = w \circ \zeta$, $\lambda = 1$ or $-1$, and $R = \mathbb{Z}_{(p)}$ for a prime $p$. Let $Q$ be a conjugation-invariant subset of $G(2)$ such that $w(g) = -\lambda$ for all $g \in Q$ and set $\hat{Q} = \hat{G}(2) \cap \zeta^{-1}(Q)$. If the order of $\pi$ is prime to $p$, then the canonical homomorphism $\rho : \Gamma_{\lambda}(\mathcal{F}) \to W_0^\lambda(R[G], (Q)_R)$ is an isomorphism, where $\mathcal{F}$ is the canonical homomorphism

$$(R[\hat{G}], -\hat{w}, \lambda, (\hat{Q})_R) \to (R[G], -w, \lambda, (Q)_R).$$
Reference