On The Smith Equivalent Representations of Oliver Groups

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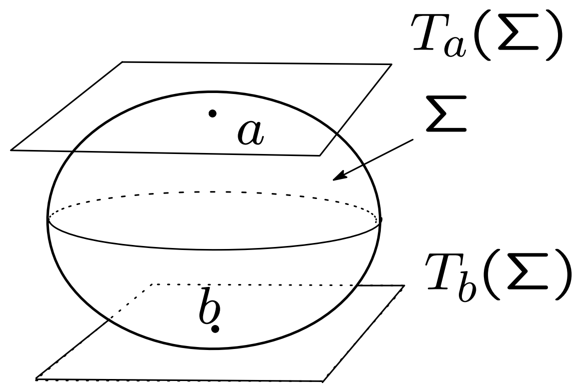
§ Problems

G: finite group.

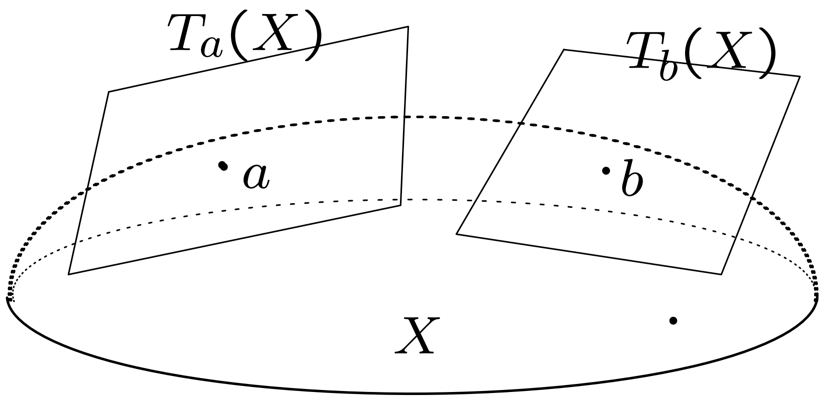
Smith Problem. Let Σ be a sphere with a G-action such that $\Sigma^G = \{a, b\}$. Then P. A. Smith asked whether $T_a(\Sigma) \cong T_b(\Sigma)$ as real G-modules, where $T_a(\Sigma)$ and $T_b(\Sigma)$ are the tangent spaces at a and b in Σ, respectively.

I.e., is Neigh. of a G-diffeomorphic to Neigh. of b?



$\mathfrak{X}$: family of smooth G-manifolds.
V, W: real G-modules.

$$V \sim_{\mathfrak{X}} W \overset{\text{Def}}{\Longleftrightarrow} \begin{cases} \exists\ X \in \mathfrak{X} \text{ s.t. } V \cong T_a(X),\ W \cong T_b(X) \\ \text{for some } a,\ b \in X^G. \end{cases}$$



Set

$$\mathfrak{D} := \{\text{disks } D \text{ s.t. } |D^G| = 2\}$$
$$\mathfrak{S} := \{\text{standard spheres } S \text{ s.t. } |S^G| = 2\}$$
$$\mathfrak{S}_{\text{ht}} := \{\text{homotopy spheres } \Sigma \text{ s.t. } |\Sigma^G| = 2\}.$$

$\text{RO}(G)$ = the real rep. ring of G
$= \{[V] - [W] \mid V$ and W: real G-modules$\}$.

$$\text{RO}(G, \mathfrak{X}) \overset{\text{Def}}{=} \{[V] - [W] \in \text{RO}(G) \mid V \sim_{\mathfrak{X}} W\}.$$

$Sm(G) := \text{RO}(G, \mathfrak{S}_{\text{ht}})$ is called the **Smith set** of G.

Smith Problem. $\text{RO}(G, \mathfrak{S}_{\text{ht}}) = 0$?

Fact (1975–85). $\text{RO}(G, \mathfrak{S}_{\text{ht}}) \neq 0$ for G:
(1) [Petrie] $C_n \times C_n$ ($n = p_1 p_2 p_3 p_4$, p_i distinct odd primes).
(2) [C-S] C_{2n} ($n \geq 4$).

Define

$$\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) := \{[V] - [W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{D}} W \text{ and } V \sim_{\mathfrak{S}} W\}.$$

$$\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) \subset \mathrm{RO}(G, \mathfrak{S}) \subset \mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}).$$

General Problem. Determine $\mathrm{RO}(G, \mathfrak{X})$ in terms of the representation theory.

Our Problem. Compute $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$ for 'small' G.

Problem. $\mathrm{RO}(G, \mathfrak{S}) = \mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})$?

Problem. $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G, \mathfrak{D}) \cap \mathrm{RO}(G, \mathfrak{S})$?

Problem. Is $\mathrm{RO}(G, \mathfrak{D}\mathfrak{S})$ a subgroup of $\mathrm{RO}(G)$?

§ Notation

$\mathcal{S}(G)$: the set of all subgroups of G
$\mathcal{P}(G) := \{P \in \mathcal{S}(G) \mid |P| = p^a (p\text{: a prime, } a\text{: int. } \geq 0)\}$
$\mathcal{P}(G)_{\mathrm{odd}} := \{P \in \mathcal{P}(G) \mid |P| = \mathrm{odd}\}$
$\mathcal{N}_p(G) := \{H \trianglelefteq G \mid |G : H| = 1,\ p\}$.

Let $\mathcal{F}$, $\mathcal{G} \subset \mathcal{S}(G)$ and $\mathfrak{A} \subset \mathrm{RO}(G)$.

$$\mathfrak{A}^{\mathcal{F}} := \{[V] - [W] \in \mathfrak{A} \mid V^H = 0 = W^H \quad (\forall\ H \in \mathcal{F})\}$$
$$\mathfrak{A}_{\mathcal{G}} := \{[V] - [W] \in \mathfrak{A} \mid \mathrm{res}_H V \cong \mathrm{res}_H W \quad (\forall\ H \in \mathcal{G})\}$$
$$\mathfrak{A}^{\mathcal{F}}_{\mathcal{G}} := (\mathfrak{A}^{\mathcal{F}})_{\mathcal{G}} = \mathfrak{A}^{\mathcal{F}} \cap \mathfrak{A}_{\mathcal{G}}.$$

Fact. [Sanchez] $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) \subset \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)_{\mathrm{odd}}}$.

$V \sim_{\mathfrak{S}_{\mathrm{ht}}} W \Rightarrow$

$$\begin{cases} V^G = 0 = W^G, \\ \mathrm{res}_P V \cong \mathrm{res}_P W \quad (P \in \mathcal{P}(G)_{\mathrm{odd}}) \end{cases}$$

Fact. If $\nexists\ g \in G$ s.t. $\mathrm{ord}(g) = 8$ then

$$\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) = \mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}})_{\mathcal{P}(G)}.$$

Fact. $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) \setminus \mathrm{RO}(G, \mathfrak{S}_{\mathrm{th}})_{\mathcal{P}(G)}$ is a finite set.

§ Oliver Groups

G is an **Oliver group** $\overset{\text{Def}}{\Longleftrightarrow} \exists\ D$ (dsik) s.t. $|D^G| = 2$

$$\overset{\text{Oliver}}{\Longleftrightarrow} \nexists\ P \trianglelefteq H \trianglelefteq G \text{ s.t. } |P| = p^a,\ |G/H| = q^b,\ H/P \text{ is cyclic}$$

Fact. [Oliver] $\mathrm{RO}(G, \mathfrak{D}) = \begin{cases} \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} & (G\text{: Oliver group}) \\ 0 & (\text{otherwise}) \end{cases}$

$$\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} = \left\{ [V] - [W] \;\middle|\; \begin{array}{c} V^G = 0 = W^G \\ \mathrm{res}_P V \cong \mathrm{res}_P W \ (P \in \mathcal{P}(G)) \end{array} \right\}$$

§ Results 1

Theorem 1. $\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) \subset \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)_{\mathrm{odd}}}$.

$$V \sim_{\mathfrak{S}_{\mathrm{ht}}} W \Rightarrow \begin{cases} V^H = 0 = W^H & (|G:H| \leq 2), \\ \mathrm{res}_P V \cong \mathrm{res}_P W & (P \in \mathcal{P}(G)_{\mathrm{odd}}) \end{cases}$$

Theorem 2. If (Sylow 2-subgrp) $G_2 \lhd G$ then

$$\mathrm{RO}(G, \mathfrak{S}_{\mathrm{ht}}) \subset \mathrm{RO}(G)^{\mathcal{N}_2(G) \cup \mathcal{N}_3(G)}_{\mathcal{P}(G)_{\mathrm{odd}}}.$$

$$V \sim_{\mathfrak{S}_{\mathrm{ht}}} W \Rightarrow \begin{cases} V^H = 0 = W^H & (H \trianglelefteq G,\ |G:H| \leq 3), \\ \mathrm{res}_P V \cong \mathrm{res}_P W & (P \in \mathcal{P}(G)_{\mathrm{odd}}) \end{cases}$$

(Compare these with $\mathrm{RO}(G, \mathfrak{D}) = \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}$ or 0.)

§ Construction of Actions on Spheres

G: Oliver group, V, W: real G-modules.

Suppose $\exists\ Y$: G-action on a disk s.t.

$$Y^G = \{a, b\},\ T_a(Y) \cong V,\ T_b(Y) \cong W.$$

Then $D(Y) = Y \bigcup_\partial Y'$: double of Y is a sphere with

$$D(Y)^G = \{a, b, a', b'\}.$$

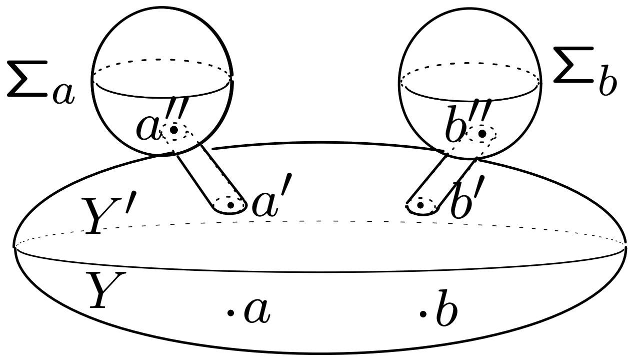
Suppose $\exists$ spheres Σ_a, Σ_b s.t.

$\Sigma_a^G = \{a''\}$, $\Sigma_b^G = \{b''\}$,

$T_{a''}(\Sigma_a) \cong T_{a'}(D(Y))$ and $T_{b''}(\Sigma_b) \cong T_{b'}(D(Y))$.

Then take the G-connected sum

$$\Sigma := D(Y) \underset{(a',a'')}{\#} \Sigma_a \underset{(b',b'')}{\#} \Sigma_b.$$



Picture of $\Sigma = D(Y) \underset{(a',a'')}{\#} \Sigma_a \underset{(b',b'')}{\#} \Sigma_b$

Clearly $\Sigma^G = \{a, b\}$. Thus $V \sim_{\mathfrak{S}} W$.

It is useful for the study of $\mathrm{RO}(G, \mathfrak{S})$ to construct

various two-fixed-point actions on disks,

one-fixed-point actions on spheres.

Oliver's Construction of Disks.

Fact. [Oliver] G: Oliver group, M: compact manifold (with trivial G-action).

If $\exists\ \xi$: real G-vect. bundle. over M s.t.

(1) $\xi^G = T(M) \oplus \varepsilon_M(\mathbb{R}^k)$,

(2) $[\mathrm{res}_{\{e\}}\xi] = 0$ in $\widetilde{KO}(\mathrm{res}_{\{e\}}M)$,

(3) $[\mathrm{res}_P\xi] = 0$ in $\widetilde{KO}_P(\mathrm{res}_P M)_{(p)} \quad (\forall\ P \in \mathcal{P}(G))$,

then $\exists$ a G-action on a disk D s.t.

$$D^G = M \text{ and } T(D)|_M \oplus \varepsilon_M(\mathbb{R}^k) \cong \xi \oplus \varepsilon_M(\mathbb{R}[G]_G{}^{\oplus m})).$$

$$\mathbb{R}[G]_G = \mathbb{R}[G] - \mathbb{R}.$$

Costruction of Spheres by G-surgery.

For each prime p, we define the **Dress group**:

$$G^{\{p\}} := \begin{cases} \text{the smallest normal subgroup } H \\ \text{s.t. } |G/H| = p^a \text{ for some } a = 0,\ 1,\ 2,\ \dots . \end{cases}$$

The set

$$\mathcal{L}(G) := \{H \in \mathcal{S}(G) \mid H \supset G^{\{p\}} \text{ for some } p\}$$

has a key role for deleting-inserting fixed point sets on **closed** G-**manifolds**.

V is a **gap** G-**module** $\overset{\text{Def}}{\Longleftrightarrow}$

$$\begin{cases} V^H = 0 & \forall\ H = G^{\{p\}} \\ \dim V^P > 2 \dim V^K & \forall (P, K) \end{cases}$$

where $P \in \mathcal{P}(G)$ and $K \in \mathcal{S}(G)$ with $K \supset P$.

G is a **gap group** $\overset{\text{Def}}{\Longleftrightarrow}$ $\exists\ V$: a gap G-module.

Theorem 3. G: Oliver group. D: a disk with G-action. Suppose the following:

(1) $\partial D^G = \emptyset$.
(2) $\partial C = \emptyset$ for $\forall$ conn. comp. C of D^H
s.t. $C^G \neq \emptyset$, where $H \in \mathcal{L}(G)$.
(3) $|\pi_1(D^P)| < \infty$ and $(|\pi_1(D^P)|, |P|) = 1$ for $\forall\ P \in \mathcal{P}(G)$.
(4) $\dim D^P \geq 5$ for $\forall\ P \in \mathcal{P}(G)$.
(5) $\dim D^P > 2(\dim D^H + 1)$ for $\forall\ P \in \mathcal{P}(G)$ and $H \supsetneq P$.
(6) $\dim D^{=H} \geq 3\ \forall\ H \in \mathcal{G}^1(G)$.

Then $\exists$ a G-action on a standard sphere S
s.t. $S^G = D^G$ and $T(S)|_{S^G} = T(D)|_{D^G}$.

$H \in \mathcal{G}^1(G) \Leftrightarrow \exists\ P \trianglelefteq H$ s.t. $P \in \mathcal{P}(H)$ and H/P is cyclic

$D^{=H} = \{x \in D \mid G_x = H\}$.

§ Results 2

Fact. [L-M] G: Oliver group s.t. $G = G^{\{2\}} \Rightarrow G$: gap group.

Fact. [Pa-So] G: gap Oliver group $\Rightarrow \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{S})$.

$$\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)} \subset \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)},$$
$$\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \bigcap_p \mathrm{RO}(G)^{\{G^{\{p\}}\}}_{\mathcal{P}(G)}.$$

Theorem 4. If G is a gap Oliver group then

$$\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) \subset \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)}.$$

$$[V] - [W] \in \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \Rightarrow V \sim_{\mathfrak{D}} W \quad \text{and} \quad V \sim_{\mathfrak{S}} W.$$

Let G be an Oliver group.

$$\begin{array}{ccccccc}
\mathrm{RO}(G,\mathfrak{D}\mathfrak{S}) & \hookrightarrow & \mathrm{RO}(G,\mathfrak{S})_{\mathcal{P}(G)} & & \hookrightarrow & & \mathrm{RO}(G,\mathfrak{D}) \\
\uparrow{\scriptstyle \text{gap}} & & \downarrow{\scriptstyle G_2 \triangleleft G} & \searrow & & & \| \\
\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} & \hookrightarrow & \mathrm{RO}(G)^{\mathcal{N}_2(G)\cup\mathcal{N}_3(G)}_{\mathcal{P}(G)} & \hookrightarrow & \mathrm{RO}(G)^{\mathcal{N}_2(G)}_{\mathcal{P}(G)} & \hookrightarrow & \mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)}
\end{array}$$

Problem. $\mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathrm{RO}(G,\mathfrak{D}\mathfrak{S})$?

We are interested in $\mathrm{RO}(G,\mathfrak{D}\mathfrak{S}) \smallsetminus \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$.

Lemma 1. Let $G = C_p$ (p: odd),
U: faithful real G-module of dim. 2, and

$$M := P(\mathbb{R} \oplus U) \quad \text{(real proj. plane)}.$$

γ: canonical line bundle over M. Then

$$\gamma^{\oplus 4} \cong_G \varepsilon_M(\mathbb{R}^4).$$

Thus

$$4[T(M)] = 0 \text{ in } \widetilde{KO}_G(M).$$

Lemma 2. Let G be an Oliver group s.t. $G/G^p = G$ if $p \neq 2$, 3, $G/G^{\{2\}} \cong C_2$, and $G/G^{\{3\}} \cong C_3$.
Let V, W be real G-modules s.t.

(1) $V^G = \mathbb{R}$ and $V^{G^{\{2\}}} = \mathbb{R}$,

(2) $V^{G^{\{3\}}} \cong \mathbb{R}[G/G^{\{3\}}]$,

(3) $W^H = 0$ if $H = G^{\{p\}}$ ($\forall\ p$),

(4) $\mathrm{res}_P V \cong_P \mathrm{res}_P W$ whenever $|P| = 2^a$.

Set $M = P(V^{G^{\{3\}}})$, and let γ denote the canonical line bundle over M and $\gamma^\perp$ the orthogonal complement (i.e., $\gamma \oplus \gamma^\perp = \varepsilon_M(V^{G^{\{3\}}})$). Then the G-vector bundle

$$\xi = (\gamma \otimes V) \oplus (\gamma^\perp \otimes W)$$

satisfies $[\mathrm{res}_P \xi] = 0$ in $\widetilde{KO}_P(\mathrm{res}_P M)_{(p)}$ for all $P \in \mathcal{P}(G)$ and $p||P|$, and moreover

$$\xi^{G^{\{3\}}} \cong_G T(M) \oplus \varepsilon_M(\mathbb{R}).$$

Lemma 3. Let G, V and W be as in Lemma 2. Then there exists a disk D with G-action having the following properties.

(1) $D^G = \{x_0\}$.
(2) Conn. comp. $D_{x_0}^{G^{\{3\}}} = P(\mathbb{R}[G/G^{\{3\}}])$.
(3) $T_{x_0}(D) \oplus \mathbb{R} \cong V \oplus (W \oplus W) \oplus \mathbb{R}[G]_{\mathcal{L}(G)}^{\oplus m}$.

where

$$\mathbb{R}[G]_{\mathcal{L}(G)} = (\mathbb{R}[G] - \mathbb{R}) - \bigoplus_p (\mathbb{R}[G/G^{\{p\}}] - \mathbb{R}).$$

Theorem 5. Suppose G is an Oliver group satisfying

(1) $G^{\{p\}} = G$ if $p \neq 3$,

(2) $G/G^{\{3\}} \cong C_3$, and

(3) $G^{\{3\}}$ has a subquotient $\cong D_{2q}$ for an odd integer $q \geq 3$.

Then $\mathrm{RO}(G)^{\{G\}}_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)}$.

Examples.

(1) $G = P\Sigma L(2, 27)$:

$G = PSL(2, 27) \rtimes C_3$, $|G| = 29484$

G is nonsolvable.

(2) $G = SG(867, 2666)$, $SG(867, 4666)$:

G are solvable.

Let $D(p, 2q) := (D_{2q} \times \cdots \times D_{2q}) \rtimes C_p$.

Theorem 6. Suppose G is an Oliver group satisfying

(1) $G^{\{p\}} = G$ if $p \neq 2$, 3,

(2) $G/G^{\{2\}} \cong C_2$,

(3) $G/G^{\{3\}} \cong C_3$, and

(4) $\exists\, N \triangleleft G$ s.t. $N \subset G^{\{2\}} \cap G^{\{3\}}$ and $G/N \cong D(3, 2q)$ for odd $q \geq 3$.

Then $\mathrm{RO}(G)^{\{G^{\{2\}}\}}_{\mathcal{P}(G)} = \mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) = \mathrm{RO}(G, \mathfrak{S})_{\mathcal{P}(G)}$. In addition,

$$\operatorname{rank} \mathrm{RO}(G)^{\{G^{\{2\}}\}}_{\mathcal{P}(G)} = \operatorname{rank} \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} + 1.$$

Hence

$$\mathrm{RO}(G, \mathfrak{D}\mathfrak{S}) \smallsetminus \mathrm{RO}(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \neq \emptyset.$$